A TOR CALCULATION

The purpose of the write-up is to present the following example for my Math 830 class.

Example. Suppose k is a field and set $R := k[X,Y]/\langle XY\rangle$. We use lower case x, y to denote images in R. Set $\mathfrak{m} := \langle x, y\rangle$ in R, so that $R/\mathfrak{m} \cong k$. We calculate $\mathrm{Tor}_n^R(R/\mathfrak{m}, R/xR)$ for all $n \geq 0$ in two different ways. We begin by noting that in R, $xf \equiv 0$ if and only if $f \in yR$ and $gy \equiv 0$ if and only if $g \in xR$.

We will first show that

$$\mathcal{F} \qquad : \cdots \xrightarrow{\phi_4} R^2 \xrightarrow{\phi_3} R^2 \xrightarrow{\phi_2} R^2 \xrightarrow{\phi_1} R \xrightarrow{\pi} R/\mathfrak{m} \to 0$$

is a free resolution of $R/\mathfrak{m}$, where

$$\begin{aligned} \phi_1 &= \begin{pmatrix} x & y \end{pmatrix} \\ \phi_2 &= \begin{pmatrix} -y & y \\ x & 0 \end{pmatrix} \\ \phi_3 &= \begin{pmatrix} y & 0 \\ y & x \end{pmatrix} \\ \phi_4 &= \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \\ \phi_5 &= \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}, \end{aligned}$$

and $\phi_n = \phi_4$, for n even and $\phi_n = \phi_5$ for n odd, if $n \geq 6$. Now, the image of each ϕ_{i+1} is contained in the kernel of ϕ_i, since the product of the matrices $\phi_i\phi_{i+1} = 0$, for all $i \geq 1$. We must check the reverse containments. To begin, if $\phi_1 \begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then $ax + by \equiv 0$ in R. Thus, $AX + BY = CXY$, in $k[X,Y]$, for some C. Therefore, $(A - CY)X + BY = 0$, so $\begin{pmatrix} A - CY \\ B \end{pmatrix} = D\begin{pmatrix} -Y \\ X \end{pmatrix}$, for some D. Thus, in $k[X,Y]$, $\begin{pmatrix} A \\ B \end{pmatrix} = D\begin{pmatrix} -Y \\ X \end{pmatrix} + C\begin{pmatrix} Y \\ 0 \end{pmatrix}$, so in R we have $\begin{pmatrix} a \\ b \end{pmatrix} \equiv d\begin{pmatrix} -y \\ x \end{pmatrix} + c\begin{pmatrix} y \\ 0 \end{pmatrix}$, showing that $\begin{pmatrix} a \\ b \end{pmatrix} \equiv \phi_2\begin{pmatrix} d \\ c \end{pmatrix}$. That is, the kernel of ϕ_1 is contained in the image of ϕ_2, which gives exactness of $\mathcal{F}$ in homological degree one.

Now suppose $\begin{pmatrix} a \\ b \end{pmatrix}$ is in the kernel of ϕ_2. Then $a\begin{pmatrix} -y \\ x \end{pmatrix} + b\begin{pmatrix} y \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ over R. Then $-ay + by \equiv 0$ and $ax \equiv 0$ in R. Thus, $-a + b \equiv cx$ and $a \equiv dy$, for some $c, d \in R$. Therefore, $b \equiv cx + dy$. Therefore, $\begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} y & 0 \\ y & x \end{pmatrix}\begin{pmatrix} d \\ c \end{pmatrix}$, showing that $\begin{pmatrix} a \\ b \end{pmatrix}$ is in the image of ϕ_2. Thus $\mathcal{F}$ is exact in homological degree two.

Now suppose $\begin{pmatrix} a \\ b \end{pmatrix}$ is in the kernel of ϕ_3. Then $a\begin{pmatrix} y \\ y \end{pmatrix} + b\begin{pmatrix} 0 \\ x \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ over R. Thus, $ay \equiv 0$ and $ay + bx \equiv 0$ in R. The first equation implies $a \equiv cx$, for some $c \in R$. Using this in the second equation we get $0 \equiv (cx)y + bx \equiv bx$, so that $b \equiv dy$, for some $d \in R$. Thus, $\begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}$, so that $\begin{pmatrix} a \\ b \end{pmatrix}$ is in the image of ϕ_4, which gives exactness of $\mathcal{F}$ in homological degree three.

Suppose $\begin{pmatrix} a \\ b \end{pmatrix}$ belongs to the kernel of ϕ_4. Then $a\begin{pmatrix} x \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ y \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $ax \equiv 0 \equiv by$ in R. Thus, $a \equiv cy$ and $b \equiv dx$, for $c, d \in R$, and hence $\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} \equiv \begin{pmatrix} a \\ b \end{pmatrix}$, showing that the kernel of ϕ_4 is contained in the image of ϕ_5, and therefore exactness holds in $\mathcal{F}$ in homological degree four. That $\mathcal{F}$ is exact now follows by the periodicity, since the remaining kernels and images have already been calculated.

Now, to calculate $\operatorname{Tor}_n^R(R/\mathfrak{m}, R/xR)$ we truncate $\mathcal{F}$ by dropping the $R/\mathfrak{m}$ term, to obtain $\tilde{\mathcal{F}}$, and tensor with R/xR. But $R/xR = k[y] = k[Y]$. For ease of notation, we set $S := R/xR$. We also note that the action of x on the R-module S is zero since its image in the ring S is zero. Thus, when we tensor $\tilde{\mathcal{F}}$ with S, we get the complex

$$\tilde{\mathcal{F}} \otimes S : \qquad \cdots \xrightarrow{\psi_4} S^2 \xrightarrow{\psi_3} S^2 \xrightarrow{\psi_2} S^2 \xrightarrow{\psi_1} S \to 0,$$

where

$$\begin{aligned}
\psi_1 &= \begin{pmatrix} 0 & y \end{pmatrix} \\
\psi_2 &= \begin{pmatrix} -y & y \\ 0 & 0 \end{pmatrix} \\
\psi_3 &= \begin{pmatrix} y & 0 \\ y & 0 \end{pmatrix} \\
\psi_4 &= \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \\
\psi_5 &= \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

and $\psi_n = \psi_4$, for n even and $\psi_n = \psi_5$ for n odd, if $n \geq 6$. Now, $\operatorname{Tor}_0^R(R/\mathfrak{m}, R/xR)$ is the cokernel of ψ_1, which is easily seen to be $S/yS \cong k$.

For $\operatorname{Tor}_1^R(R/\mathfrak{m}, R/xR)$ we first calculate the kernel of ψ_1. Working in S, if $\begin{pmatrix} a \\ b \end{pmatrix}$ is in the kernel of ψ_1, then $a0 + by \equiv 0$. Since S is an integral domain, $b = 0$, and a can be anything. Thus, the kernel of ψ_1 is $S \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If $\begin{pmatrix} c \\ d \end{pmatrix}$ is in the image of ψ_2, then $\begin{pmatrix} c \\ d \end{pmatrix} \equiv e \begin{pmatrix} -y \\ 0 \end{pmatrix} + f \begin{pmatrix} y \\ 0 \end{pmatrix}$, for $e, f \in S$. Therefore, $(e - f)y \equiv c$ and $d \equiv 0$ in S. This shows that c can be any element in yS and $d \equiv 0$. Thus, the image of ψ_2 equals $S \cdot \begin{pmatrix} y \\ 0 \end{pmatrix}$. Therefore, we have $\operatorname{Tor}_1^R(R/\mathfrak{m}, R/xR) = S \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} / S \cdot \begin{pmatrix} y \\ 0 \end{pmatrix} \cong S/yS = k$.

One more calculation for this case. For $\operatorname{Tor}_2^R(R/\mathfrak{m}, R/xR)$, we first calculate the kernel of ψ_2. If $\begin{pmatrix} a \\ b \end{pmatrix}$ is in the kernel of ψ_2, then $a \begin{pmatrix} -y \\ 0 \end{pmatrix} + b \begin{pmatrix} y \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ over S. Thus, $(-a + b)y \equiv 0$ in S, so $-a + b \equiv 0$, i.e., $a \equiv b$ in S. Thus, the kernel of ψ_2 is $S \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. If $\begin{pmatrix} c \\ d \end{pmatrix}$ is in the image of ψ_3, then $\begin{pmatrix} c \\ d \end{pmatrix} \equiv e \begin{pmatrix} y \\ y \end{pmatrix} + f \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, for $e, f \in S$. This shows that the image of ψ_3 is $S \cdot \begin{pmatrix} y \\ y \end{pmatrix}$. Thus, $\operatorname{Tor}_2^R(R/\mathfrak{m}, R/xR) = S \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} / S \cdot \begin{pmatrix} y \\ y \end{pmatrix} \cong S/yS = k$. Continuing, with one more calculation, and using the periodicity of $\tilde{\mathcal{F}}$, we have that $\operatorname{Tor}_n^R(R/\mathfrak{m}, R/xR) \cong k$, for all $n \geq 0$.

We now calulate $\operatorname{Tor}_n^R(R/\mathfrak{m}, R/xR)$ by taking a projective resolution of R/xR over R and tensoring it with $R/\mathfrak{m}$. We'll see that this is a much easier calculation. We clearly have the following free resolution of R/xR over R:

$$\cdots \xrightarrow{\cdot x} R \xrightarrow{\cdot y} R \xrightarrow{\cdot x} R \xrightarrow{\cdot y} R \xrightarrow{\cdot x} R \to R/xR \to 0.$$

Dropping R/xR and tensoring with $R/\mathfrak{m}$ we obtain the complex

$$\cdots \xrightarrow{\cdot x} R/\mathfrak{m} \xrightarrow{\cdot y} R/\mathfrak{m} \xrightarrow{\cdot x} R/\mathfrak{m} \xrightarrow{\cdot y} R/\mathfrak{m} \xrightarrow{\cdot x} R/\mathfrak{m} \to 0.$$

Since $x, y \in \mathfrak{m}$, the kernel in each homological degree is $R/\mathfrak{m}$ and the image in the same degree is 0. Thus, $\operatorname{Tor}_n^R(R/\mathfrak{m}, R/xR) \cong R/\mathfrak{m} = k$, for all $n \geq 0$.